Fort Street
High School

NESA Number:

Name:

2024 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Approved calculators may be used
- A reference sheet is provided
- Marks may be deducted for careless or badly arranged work.
- In Questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks : 70 **Section I – 10 marks**

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks

- Allow about 1 hours and 45 minutes for this section
- Write your student number on each answer booklet.
- Attempt Questions 11 – 15

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1 – 10.

---

1. Given $f(x) = 1 + \sqrt{x}$, the domain and range of $f^{-1}(x)$ are:

A. Domain: $\{x \geq 0\}$, Range: $\{y \geq 0\}$

B. Domain: $\{x \geq 0\}$, Range: $\{y \geq 1\}$

C. Domain: $\{x \geq 1\}$, Range: $\{y \geq 0\}$

D. Domain: $\{x \geq 1\}$, Range: $\{y \geq 1\}$

2. The acute angle (to the nearest degree) between the vectors $\underset{\sim}{u} = \begin{pmatrix} \sqrt{3} \\ 2 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 9 \\ \sqrt{17} \end{pmatrix}$ is:

A. $24°$

B. $51°$

C. $46°$

D. $48°$

3. The solution to the inequality $\dfrac{3}{x-2} \leq 4$ is given by:

A. $x < -2$ and $x \geq -\dfrac{11}{4}$

B. $x > -2$ and $x \leq -\dfrac{11}{4}$

C. $x < 2$ and $x \geq \dfrac{11}{4}$

D. $x > 2$ and $x \leq \dfrac{11}{4}$

4. A function is defined as $f(x) = \tan^{-1}(\tan(x))$.

The value of $f\left(\frac{9\pi}{4}\right)$ is:

A. $\frac{\pi}{4}$

B. $\frac{5\pi}{4}$

C. $\frac{7\pi}{4}$

D. $\frac{9\pi}{4}$

5. When the polynomial $P(x)$ is divided by $x+1$ it gives a remainder of n^2, where n is a positive integer. When $P(x)$ is divided by $(x+1)(x+3)$ it gives a remainder $(nx+6)$. The value of n is:

A. 1

B. 3

C. 6

D. 2

6. The $\int \sin 3x \sin x \, dx$ is equal to:

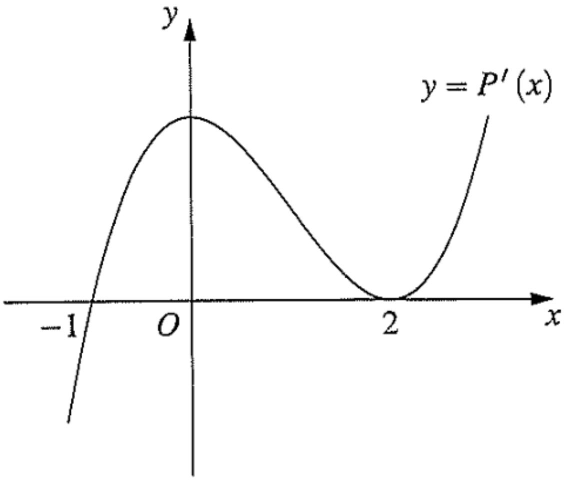
A. $\frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + C$

B. $\frac{1}{8}\sin 4x - \frac{1}{4}\sin 2x + C$

C. $\frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x + C$

D. $\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C$

7. In the graph below, $y = P'(x)$ represents the first derivative of a polynomial $P(x)$ of degree 4, where $P(x)$ has a multiple root.



Which of the following could be true about the polynomial $P(x)$?

A. $x = -1$ is a root of multiplicity 3.

B. $x = 0$ is a root of multiplicity 2.

C. $x = 2$ is a root of multiplicity 2.

D. $x = 2$ is a root of multiplicity 3.

8. The derivative of $f(x) = 2x^2 \cos^{-1}(2x)$ is:

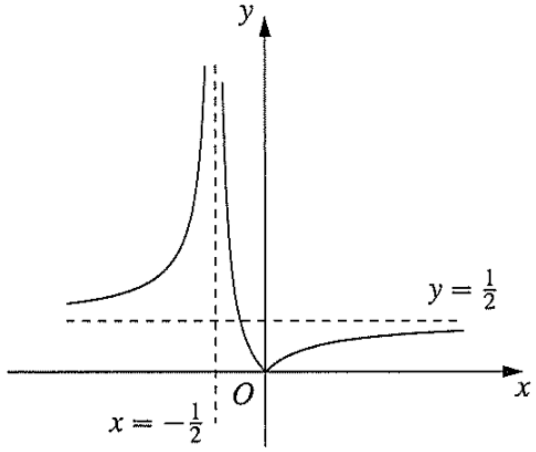
A. $\dfrac{-8x}{\sqrt{1-2x^2}}$

B. $\dfrac{-8x}{\sqrt{1-4x^2}}$

C. $\dfrac{-4x^2}{\sqrt{1-2x^2}} + 4x\cos^{-1} 2x$

D. $\dfrac{-4x^2}{\sqrt{1-4x^2}} + 4x\cos^{-1} 2x$

9. The diagram below shows the graph of the function $y = g(x)$, which is the result of a set of transformations on the graph of $y = f(x)$.



If $f(x) = \dfrac{1}{x-2} + 2$, which equation best represents $g(x)$?

A. $g(x) = \dfrac{1}{|f(x-2)|}$

B. $g(x) = |f(x-2)|$

C. $g(x) = \dfrac{1}{|f(x+2)|}$

D. $g(x) = |f(x+2)|$

10. Vector $\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is projected on to vector $\underset{\sim}{b} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$. Given that $\text{Proj}_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{1}{2}\underset{\sim}{b}$, which of the following could possibly be vector $\underset{\sim}{a}$?

A. $\underset{\sim}{a} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$

B. $\underset{\sim}{a} = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$

C. $\underset{\sim}{a} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

D. $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Section II

60 marks
Attempt Questions 11 – 15
Allow about 1 hour and 45 minutes for this section
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.
In questions 11 – 15, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (13 marks) Use the Question 11 Writing Booklet

(a) Let P be the point $(12,5)$. Find the position vector of point P, and hence find a unit vector in the direction of the position vector. **2**

(b) Find the exact value of $\int \sin^2 \frac{2}{3}x\,dx$. **1**

(c) Use the t- identities to solve the equation $2\sin x + \cos x + 1 = 0$ for $0 \le x \le 2\pi$. **4**

(d) Find $\int \frac{1}{\sqrt{64-49x^2}}\,dx$. **2**

(e) (i) Express $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x+\alpha)$ for $R>0$ and α acute. **2**

(ii) Hence, solve $\cos x - \sqrt{3}\sin x = 2$ for $0 \le x \le 2\pi$. **2**

End of Question 11

Question 12 (12 marks) Use the Question 12 Writing Booklet

(a) There are four teams consisting of two tennis players in each team. **2**

They are seated around a circular table.

In how many ways can the teams be arranged around the table if the players in each team are to stay together.

(b) By using the substitution $x = \sin\theta$, find $\displaystyle\int_0^{\frac{1}{2}} \left(1-x^2\right)^{-\frac{3}{2}} dx$ **3**

(c) Prove by mathematical induction that $3^{3n-1} + 5^{3n-2}$ is divisible by 7, **4**

for any integer $n \geq 1$.

(d) A cake tin cools such that its temperature T degrees, t minutes after it is removed **3**

from the oven is given by:

$T = R + Ae^{-kt}$ Where k, R and A are constants

It is known that:

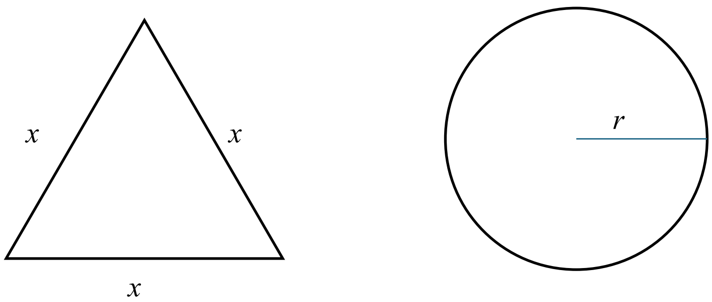
$\dfrac{dT}{dt} = -k\left(T - R\right)$ Do NOT prove this.

When removed from an oven the cake tin has a temperature of $180°C$. If the cake tin takes one minute to cool to $150°C$ and the room temperature is $20°C$, find the time to the nearest minute it takes for the cake tin to cool to $80°C$. (Assume that the cake tin cools at a rate proportional to the difference between the temperature of the cake tin and the temperature of the surrounding air.)

End of Question 12

Question 13 (13 marks) Use the Question 13 Writing Booklet

(a) The side lengths of an equilateral triangle are shrinking at a rate of $\sqrt{x}\ m/s$, where x metres is the length of each side of the triangle. The triangle and the circle shown below always have the same area A.



(i) Show that the rate of change of the area of the triangle is: **2**

$$\frac{-\sqrt{3}\,x\sqrt{x}}{2}\quad \mathrm{m^2/s}$$

(ii) Find the rate of change of the radius of the circle when $x = 5$ metres. **3**

Give the answer correct to 2 decimal places.

(b) By considering $(1+x)^{2n} + (1-x)^{2n}$, where n is a positive integer, show that: **3**

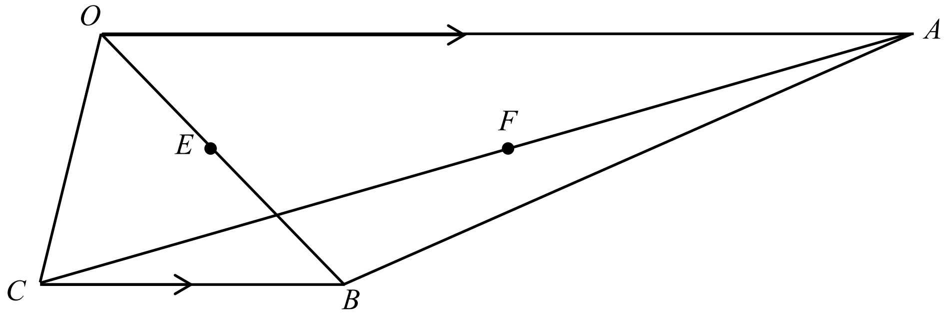
$$1+\binom{2n}{2}+\binom{2n}{4}+\ldots+\binom{2n}{2n}=\frac{4^n}{2}.$$

Question 13 continues on the next page

Question 13 continued

(c) $OABC$ is trapezium with $\overrightarrow{OA} = 3\overrightarrow{CB}$. $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

E and F are midpoints of OB and CA respectively.



(i) Show that $\overrightarrow{OE} = \frac{1}{2}\left(\underset{\sim}{c} + \frac{1}{3}\underset{\sim}{a}\right)$. **1**

(ii) Find $\overrightarrow{EF}$. **3**

(iii) Hence, prove that $CEFB$ is a parallelogram. **1**

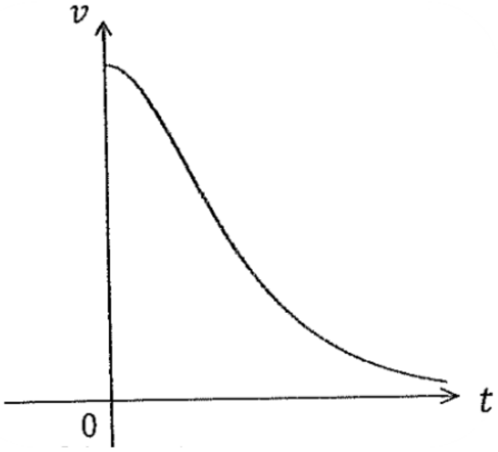
End of Question 13

Question 14 (11 marks) Use the Question 14 Writing Booklet

(a) A particle P is moving along the x axis. At time $t = 0$, it was at $x = 0$. Its velocity **4**

$v\ ms^{-1}$ at a time t is given by $v = \dfrac{e^t}{1+(e^t)^2}$.

The graph below shows its velocity as a function of time.



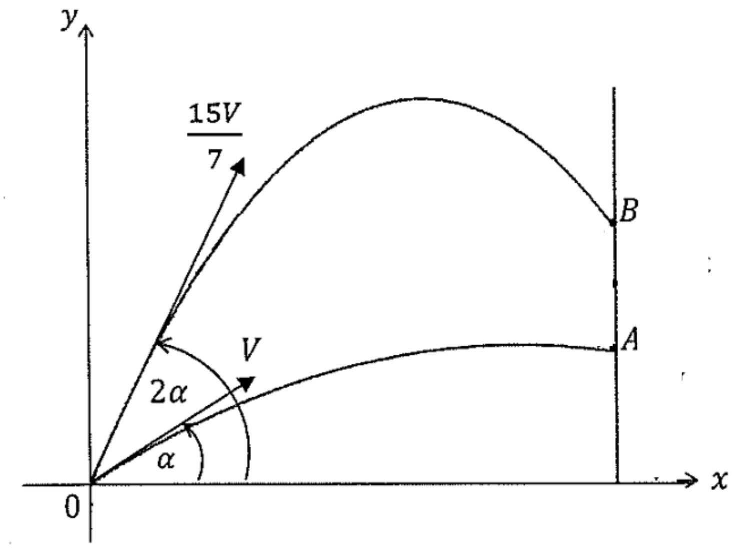
Given that the displacement of the particle when $t = k$ seconds is $\tan^{-1}\dfrac{1}{2}$ metres,

show that $k = \ln 3$.

Question 14 continues on the next page.

Question 14 continued

(b) Jacob hits two balls at the same time from the origin O towards a target on a vertical wall. The velocity of the first ball is $V\,ms^{-1}$ and its angle of projection is α to the horizontal while the velocity of the second ball is $\dfrac{15V}{7}\,ms^{-1}$ and its angle of projection to the horizontal is 2α.



(i) Show that the position vector of the second ball t seconds after being projected is given by: **3**

$$\underset{\sim}{r_2}(t)=\begin{pmatrix}\dfrac{15V}{7}t\cos 2\alpha\\ -\dfrac{1}{2}gt^2+\dfrac{15V}{7}t\sin 2\alpha\end{pmatrix}$$

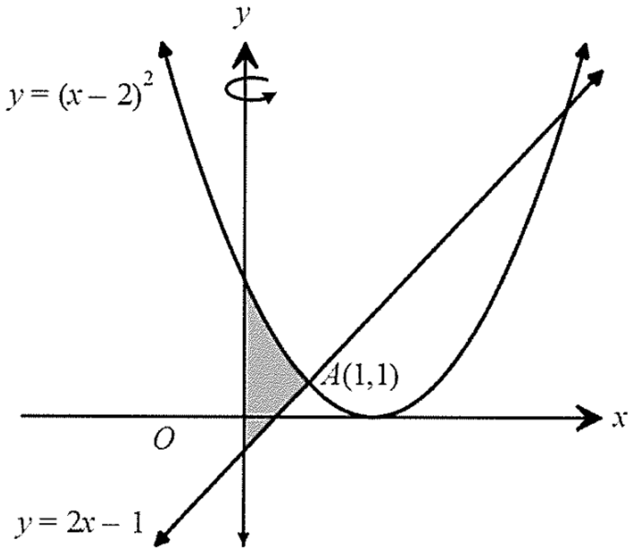
(ii) It is known that both balls hit the wall after T seconds. **4**

Show that $\cos\alpha=\dfrac{5}{6}$.

End of Question 14

Question 15 (11 marks) Use the Question 15 Writing Booklet

(a) Consider the shaded region bounded by the line $y = 2x - 1$, the curve $y = (x-2)^2$ and the y-axis. **4**



Find the volume of the solid of revolution formed by rotating this region about the y-axis.

(b) (i) Show that the solution of $\cos 3x - \sin 2x = 0$, for $0 < x < \frac{\pi}{2}$ is given by **3**

$$\sin x = \frac{\sqrt{5}-1}{4}.$$

You may use the identity $\cos 3x = 4\cos^3 x - 3\cos x$. DO NOT PROVE THIS.

(ii) Without a calculator, verify that $x = \frac{\pi}{10}$ is a solution to $\cos 3x = \sin 2x$. **1**

(iii) Using the results obtained in parts (i) and (ii) prove that **3**

$$\sin\frac{\pi}{5}\cos\frac{\pi}{10} = \frac{\sqrt{5}}{4}.$$

END OF EXAMINATION

Fort Street
High School

Solutions

2024

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

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Total marks : 70

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- Allow about 15 minutes for this section

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A. Domain: $\{x \geq 0\}$, Range: $\{y \geq 0\}$

B. Domain: $\{x \geq 0\}$, Range: $\{y \geq 1\}$

C. Domain: $\{x \geq 1\}$, Range: $\{y \geq 0\}$

D. Domain: $\{x \geq 1\}$, Range: $\{y \geq 1\}$

$D_f : x \geqslant 0 \quad R_f : y \geqslant 1$

$D_{f^{-1}} : x \geqslant 1 \quad R_{f^{-1}} : y \geqslant 0$

2. The acute angle (to the nearest degree) between the vectors $\underset{\sim}{u} = \begin{pmatrix} \sqrt{3} \\ 2 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 9 \\ \sqrt{17} \end{pmatrix}$ is:

A. $24°$

B. $51°$

C. $46°$

D. $48°$

$$\cos\theta = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$$
$$= \frac{(9\sqrt{3} + 2\sqrt{17})}{\sqrt{7} \times \sqrt{98}}$$

$\therefore \theta \doteqdot 24°29'34''$ closer to $24°$.

3. The solution to the inequality $\dfrac{3}{x-2} \leq 4$ is given by:

A. $x < -2$ and $x \geq -\dfrac{11}{4}$

B. $x > -2$ and $x \leq -\dfrac{11}{4}$

C. $x < 2$ and $x \geq \dfrac{11}{4}$

D. $x > 2$ and $x \leq \dfrac{11}{4}$

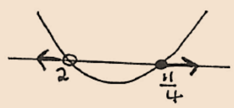
$3(x-2) \leq 4(x-2)^2 \quad , \quad x \neq 2$

$4(x-2)^2 \geqslant 3(x-2)$

$4(x-2)^2 - 3(x-2) \geqslant 0$

$(x-2)(4x-8-3) \geqslant 0$

$(x-2)(4x-11) \geqslant 0$



4. A function is defined as $f(x) = \tan^{-1}(\tan(x))$.

The value of $f\left(\frac{9\pi}{4}\right)$ is:

A. $\frac{\pi}{4}$

B. $\frac{5\pi}{4}$

C. $\frac{7\pi}{4}$

D. $\frac{9\pi}{4}$

$$f\left(\tfrac{9\pi}{4}\right) = \tan^{-1}\left(\tan\left(\tfrac{9\pi}{4}\right)\right)$$
$$= \tan^{-1}\left(\tan\left(2\pi + \tfrac{\pi}{4}\right)\right)$$
$$= \tan^{-1}\left(\tan\tfrac{\pi}{4}\right)$$
$$= \tfrac{\pi}{4}$$

5. When the polynomial $P(x)$ is divided by $x+1$ it gives a remainder of n^2, where n is a positive integer. When $P(x)$ is divided by $(x+1)(x+3)$ it gives a remainder $(nx+6)$. The value of n is:

A. 1

B. 3

C. 6

D. 2

$P(-1) = n^2$

$P(x) = (x+1)(x+3)Q(x) + (nx+6)$

$P(-1) = -n+6$

$n^2 = -n+6$

$n^2 + n - 6 = 0$

$(n+3)(n-2) = 0 \quad \therefore n = 2$, given $n =$ positive integer

6. The $\int \sin 3x \sin x \, dx$ is equal to:

A. $\frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x + C$

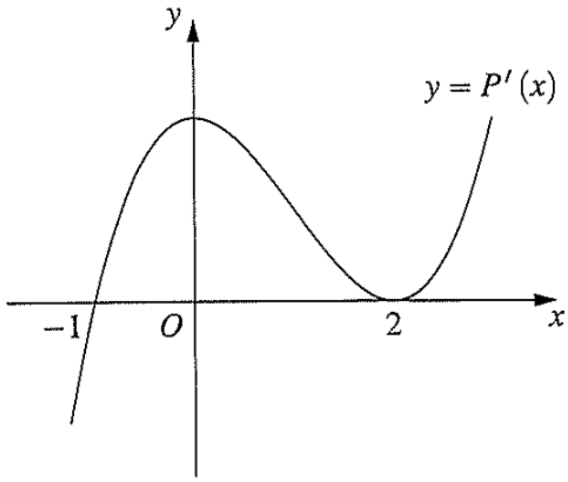
B. $\frac{1}{8}\sin 4x - \frac{1}{4}\sin 2x + C$

C. $\frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x + C$

D. $\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C$

$$\int \sin 3x \sin x \, dx$$
$$= \tfrac{1}{2}\int (\cos 2x - \cos 4x)\, dx$$
$$= \tfrac{1}{2}\left[\tfrac{1}{2}\sin 2x - \tfrac{1}{4}\sin 4x\right] + C$$
$$= \tfrac{1}{4}\sin 2x - \tfrac{1}{8}\sin 4x + C$$

7. In the graph below, $y = P'(x)$ represents the first derivative of a polynomial $P(x)$ of degree 4, where $P(x)$ has a multiple root.



Which of the following could be true about the polynomial $P(x)$?

A. $x = -1$ is a root of multiplicity 3.

B. $x = 0$ is a root of multiplicity 2.

C. $x = 2$ is a root of multiplicity 2.

D. $x = 2$ is a root of multiplicity 3.

8. The derivative of $f(x) = 2x^2 \cos^{-1}(2x)$ is:

A. $\dfrac{-8x}{\sqrt{1-2x^2}}$

B. $\dfrac{-8x}{\sqrt{1-4x^2}}$

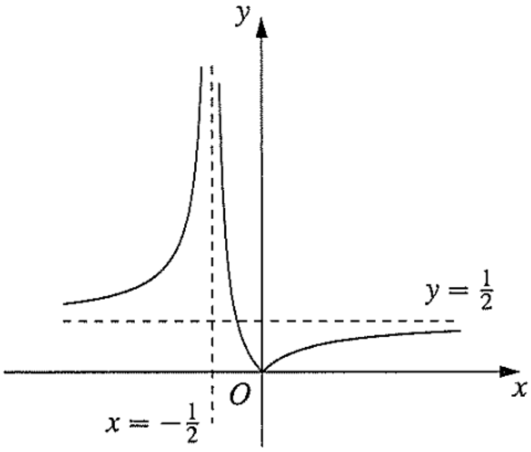
C. $\dfrac{-4x^2}{\sqrt{1-2x^2}} + 4x\cos^{-1} 2x$

D. $\dfrac{-4x^2}{\sqrt{1-4x^2}} + 4x\cos^{-1} 2x$

$$f'(x) = \left[\cos^{-1}(2x)\right]4x - 2x^2 \cdot \frac{2}{\sqrt{1-4x^2}}$$

$$= -\frac{4x^2}{\sqrt{1-4x^2}} + 4x\cos^{-1}(2x)$$

9. The diagram below shows the graph of the function $y = g(x)$, which is the result of a set of transformations on the graph of $y = f(x)$.



If $f(x) = \dfrac{1}{x-2} + 2$, which equation best represents $g(x)$?

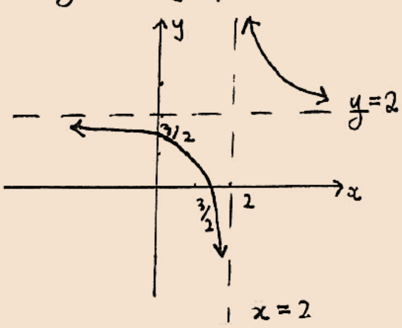
A. $g(x) = \dfrac{1}{|f(x-2)|}$

B. $g(x) = |f(x-2)|$

C. $g(x) = \dfrac{1}{|f(x+2)|}$

D. $g(x) = |f(x+2)|$

$y = f(x)$ graph



So, $y = g(x)$ is the reciprocal function, shifted 2 units left, and has all negative y-values reflected above the x-axis.

10. Vector $\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is projected on to vector $\underset{\sim}{b} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$. Given that $\text{Proj}_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{1}{2}\underset{\sim}{b}$, which of the following could possibly be vector $\underset{\sim}{a}$?

A. $\underset{\sim}{a} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$

B. $\underset{\sim}{a} = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$

C. $\underset{\sim}{a} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

D. $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$\frac{1}{2}\underset{\sim}{b} = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2} \times \underset{\sim}{b}$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \dfrac{4a_1 + 6a_2}{52} \times \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \dfrac{2a_1 + 3a_2}{\cancel{26}\ 13} \times \overset{1}{\cancel{2}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \dfrac{10 + 3}{13} \times \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (substitute answers, 'C' works)

6

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Section II

60 marks
Attempt Questions 11 – 15
Allow about 1 hour and 45 minutes for this section
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.
In questions 11 – 15, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (13 marks) Use the Question 11 Writing Booklet

(a) Let P be the point $(12,5)$. Find the position vector of point P, and hence find a unit vector in the direction of the position vector. **2**

(a) $P(12,5)$, position vector $12\underset{\sim}{i}+5\underset{\sim}{j}$ ✓ (① for vector)

Unit vector $= \dfrac{\overrightarrow{OP}}{|\overrightarrow{OP}|}$

$= \dfrac{12\underset{\sim}{i}+5\underset{\sim}{j}}{\sqrt{144+25}}$

① for either answer

$= \dfrac{1}{13}(12\underset{\sim}{i}+5\underset{\sim}{j})$

(b) Find the exact value of $\int \sin^2 \frac{2}{3}x\,dx$. **1**

$\int \sin^2\left(\frac{2}{3}x\right)dx$

$= \frac{1}{2}\int\left(1-\cos\frac{4}{3}x\right)dx$

$= \frac{1}{2}\left[x - \frac{3}{4}\sin\frac{4}{3}x\right]+C$ ✓ ① for correct primitive function
Do not deduct mark if C is missing

(c) Use the t - identities to solve the equation $2\sin x + \cos x + 1 = 0$ for $0 \le x \le 2\pi$. **4**

(c) $2\sin x + \cos x + 1 = 0$

$$\frac{4t}{1+t^2} + \frac{1-t^2}{1+t^2} + 1 = 0$$

$$\frac{4t + 1 - t^2 + 1 + t^2}{1+t^2} = 0$$

$$\frac{4t+2}{1+t^2} = 0$$ ✓

$\boxed{\text{① for some progress made after substituting } t \text{ results}}$

$4t + 2 = 0$

$\therefore t = -\frac{1}{2}$ ✓ $\boxed{\text{① for } t \text{ value}}$

$\tan\frac{x}{2} = -\frac{1}{2}$, $0 \le \frac{x}{2} \le \pi$

related $\angle \doteqdot 0.4636^c$

(accept related $\angle = \tan^{-1}(\frac{1}{2})$)

$\therefore \frac{x}{2} = \pi - 0.4636^c$ or $\frac{x}{2} = \pi - \tan^{-1}(\frac{1}{2})$

$x = 2(\pi - 0.4636^c)$

$x \doteqdot 5.3559^c$ or accept $x = 2(\pi - \tan^{-1}(\frac{1}{2}))$

Test $x = \pi$

$2\sin\pi + \cos\pi + 1 = 0$

$0 - 1 + 1 = 0$

$0 = 0$

$\therefore x = \pi,\ 5.3559^c$ (4 d.p.) ✓✓

$2(\pi - \tan^{-1}\frac{1}{2})$

$\boxed{\text{② marks One for each answer.}}$

(d) Find $\int \frac{1}{\sqrt{64-49x^2}}\,dx$. **2**

(d) $\int \frac{1}{\sqrt{64-49x^2}}\,dx$

$= \int \frac{1}{\sqrt{8^2-(7x)^2}}$

$= \frac{1}{7}\int \frac{7}{\sqrt{8^2-(7x)^2}}$ ✓ $\boxed{\text{1 for showing working}}$

$= \frac{1}{7}\sin^{-1}\left(\frac{7x}{8}\right)+C$ ✓ $\boxed{\text{1 correct primitive function}}$

(e) (i) Express $\cos x-\sqrt{3}\sin x$ in the form $R\cos(x+\alpha)$ for $R>0$ and α acute. **2**

$\cos x-\sqrt{3}\sin x = R\cos(x+\alpha)$

$= R[\cos x\cos\alpha - \sin x\sin\alpha]$

$\cos x-\sqrt{3}\sin x = R\cos x\cos\alpha - R\sin x\sin\alpha$

$R\cos\alpha = 1, \quad R\sin\alpha=\sqrt{3}$

$$\left.\begin{aligned} &R^2\cos^2\alpha + R^2\sin^2\alpha = 1+3 \\ &R^2(1)=4 \\ &\therefore R=2 \quad (R>0) \end{aligned}\right\} \checkmark \ \boxed{\text{1 for R with working}}$$

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{\sqrt{3}}{1}$

$\tan\alpha = \sqrt{3}$

$\therefore \alpha = \frac{\pi}{3}, \quad \alpha = \text{acute} \angle$ ✓ $\boxed{\text{1 for } \alpha \text{ with working}}$

$\therefore \cos x-\sqrt{3}\sin x = 2\cos\left(x+\frac{\pi}{3}\right)$

(ii) Hence, solve $\cos x - \sqrt{3}\sin x = 2$ for $0 \le x \le 2\pi$. **2**

f) ii) $\cos x - \sqrt{3}\sin x = 2$, $0 \le x \le 2\pi$

$2\cos\left(x + \frac{\pi}{3}\right) = 2$

① for trig equation rewritten in alternate form ✓

$\cos\left(x + \frac{\pi}{3}\right) = 1$, $\frac{\pi}{3} \le x + \frac{\pi}{3} \le \frac{7\pi}{3}$

$x + \frac{\pi}{3} = 2\pi$

$\therefore x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ ✓

① for solution

End of Question 11

Question 12 (12 marks) Use the Question 12 Writing Booklet

(a) There are four teams consisting of two tennis players in each team. **2**

They are seated around a circular table.

In how many ways can the teams be arranged around the table if the players in each team are to stay together.

Number of ways

$= (4-1)! \times (2!)^4$ ① for number of ways around a circle ① for arranging pairs $(2!)^4$

$= 3!(2!)^4$ or 96 Accept any version for answer

(b) By using the substitution $x = \sin\theta$, find $\int_0^{\frac{1}{2}} \left(1-x^2\right)^{-\frac{3}{2}} dx$ **3**

$x = \sin\theta$, $dx = \cos\theta \, d\theta$

$x = 0 \rightarrow \theta = 0$

$x = \frac{1}{2} \rightarrow \theta = \frac{\pi}{6}$

$\int_0^{1/2} (1-x^2)^{-3/2} \, dx$

$= \int_0^{\pi/6} (1-\sin^2\theta)^{-3/2} . \cos\theta \, d\theta$ ✓ ① rewriting integral correctly in terms of θ

$= \int_0^{\pi/6} \frac{1}{\left(\sqrt{\cos^2\theta}\right)^3} \cos\theta \, d\theta$

$= \int_0^{\pi/6} \frac{1}{\cos^3\theta} . \cos\theta \, d\theta$

$= \int_0^{\pi/6} \sec^2\theta \, d\theta$ ✓ ① for progress

$= \left[\tan\theta\right]_0^{\pi/6}$

$= \tan\frac{\pi}{6} - \tan 0$

$= \frac{1}{\sqrt{3}}$ ✓ ① for answer

(c) Prove by mathematical induction that $3^{3n-1}+5^{3n-2}$ is divisible by 7, **4**

for any integer $n \geq 1$.

(c) Test $n=1$

$3^{3n-1}+5^{3n-2}$

$= 3^2 + 5^1$

$= 14$ which is divisible by 7 ✓ ① for testing $n=1$

$\therefore$ statement is true for $n=1$

Assume true for $n=k$, k = +ve integer

i.e. $3^{3k-1}+5^{3k-2} = 7P$, P = some integer

And prove true for $n=k+1$

$3^{3(k+1)-1} + 5^{3(k+1)-2}$

$= 3^{3k+2} + 5^{3k+1}$

$= 3^{3k-1+3} + 5^{3k+1}$

$= (7P - 5^{3k-2})3^3 + 5^{3k+1}$ using assumption ✓ ① for using assumption

$= 27(7P - 5^{3k-2}) + 5^{3k-2+3}$

$= 189P - 27(5^{3k-2}) + 5^{3k-2}(5^3)$ ✓ ① For correct progress

$= 189P + 98(5^{3k-2})$

$= 7\left[27P + 14(5^{3k-2})\right]$ ① for showing divisible by 7 when $n=k+1$

which is divisible by 7

$\therefore$ statement is true for $n=k+1$, assuming true for $n=k$

So by mathematical induction, statement is true for all integers $n \geq 1$. ✓

(d) A cake tin cools such that its temperature T degrees, t minutes after it is removed from the oven is given by: $T = R + Ae^{-kt}$ Where k, R and A are constants **3**

It is known that: $\dfrac{dT}{dt} = -k\left(T - R\right)$ Do NOT prove this.

When removed from an oven the cake tin has a temperature of $180°C$. If the cake tin takes one minute to cool to $150°C$ and the room temperature is $20°C$, find the time to the nearest minute it takes for the cake tin to cool to $80°C$. (Assume that the cake tin cools at a rate proportional to the difference between the temperature of the cake tin and the temperature of the surrounding air.)

(d) $T = R + Ae^{-kt}$, $\dfrac{dT}{dt} = -k(T-R)$

at $t=0$, $T = 180°C$ ①

$t=1$, $T=150°C$ ②

$t=?$ $T=80°C$ ③

$T = R + Ae^{-kt}$, $R = 20$

$T = 20 + Ae^{-kt}$ (sub R)

$180 = 20 + Ae^{0}$ (sub ①)

$160 = A$ ✓ $\boxed{\text{① for A}}$

$\therefore T = 20 + 160e^{-kt}$

$150 = 20 + 160e^{-k}$ (sub②)

$130 = 160e^{-k}$

$e^{-k} = \frac{13}{16}$

$\log_e\left(\frac{13}{16}\right) = -k$

$\therefore k = -\ln\left(\frac{13}{16}\right) = \ln\left(\frac{16}{13}\right)$ ✓ $\boxed{\text{① for k}}$

$\therefore T = 20 + 160e^{[\ln(13/16)]t}$

$80 = 20 + 160e^{[\ln(13/16)]t}$ sub③

$60 = 160e^{[\ln(13/16)]t}$

$\frac{3}{8} = e^{[\ln(13/16)]t}$

$\therefore \ln(3/8) = \ln(13/16).t$

$\therefore t = \dfrac{\ln(3/8)}{\ln(13/16)}$

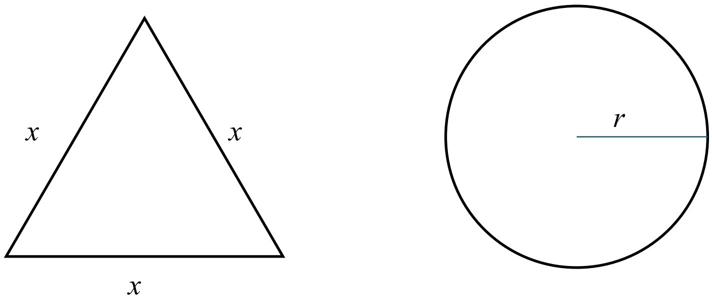
$\doteqdot 4.7237$ ✓ $\boxed{\text{① for correct solution. Ignore rounding errors but make a comment}}$

$\therefore$ It takes approx. 5 minutes for cake tin to cool to 80°C.

End of Question 12

Question 13 (13 marks) Use the Question 13 Writing Booklet

(a) The side lengths of an equilateral triangle are shrinking at a rate of $\sqrt{x}\ m/s$, where x metres is the length of each side of the triangle. The triangle and the circle shown below always have the same area A.



(i) Show that the rate of change of the area of the triangle is: **2**

$$\frac{-\sqrt{3}\,x\sqrt{x}}{2} \quad \mathrm{m^2/s}$$

(a) i) $A = \frac{1}{2} \times x \times x \times \sin 60^\circ$

$= \frac{1}{2} \times x^2 \times \frac{\sqrt{3}}{2}$

$A = \frac{\sqrt{3}}{4} x^2$ ✓ $\boxed{\text{① for area of triangle}}$

$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

$= \frac{\cancel{2}^{1}\sqrt{3}\,x}{\cancel{4}_{2}} \times -\sqrt{x}$ ✓ $\boxed{\text{① for working}}$

$= \frac{-\sqrt{3}\,x\sqrt{x}}{2}$

(ii) Find the rate of change of the radius of the circle when $x = 5$ metres. **3**

Give the answer correct to 2 decimal places.

ii) Circle: $\frac{dr}{dt} = ?$ when $x = 5$

$A = \pi r^2$

$\frac{\sqrt{3}}{4}x^2 = \pi r^2$

$r = \sqrt{\frac{\sqrt{3}x^2}{4\pi}} = \left(\frac{\sqrt{3}x^2}{4\pi}\right)^{1/2}$

$r = \frac{3^{1/4}}{2\sqrt{\pi}} \cdot x$ $\boxed{\text{① for attempting to find } r}$

$\frac{dr}{dt} = \frac{dr}{dx} \times \frac{dx}{dt}$

$= \frac{3^{1/4}}{2\sqrt{\pi}} \times -\sqrt{x}$ ✓ $\boxed{\text{① for working to find } \frac{dr}{dt}}$

at $x=5$ $\frac{dr}{dt} = \frac{3^{1/4}}{2\sqrt{\pi}} \times -\sqrt{5}$

$= -0.83$ m/s (2 d.p.)

✓ $\boxed{\text{① for answer. If left exact, ignore \& make comment}}$

$\frac{dr}{dt} = ?$ when $x = 5$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$\frac{-\sqrt{3}x\sqrt{x}}{2} = 2\pi r \times \frac{dr}{dt}$

$\frac{-\sqrt{3}x^{1.5}}{2} = \frac{2 \times \pi \times 3^{1/4} x}{2\sqrt{\pi}} \times \frac{dr}{dt}$.

$\frac{dr}{dt} = \frac{-\sqrt{3}x^{1.5} \times 2 \times \sqrt{\pi}}{2 \times 2 \times \pi \times 3^{1/4} \times 5}$ $x=5$

$= -0.83$ m/s (2 d.p.)

(b) By considering $(1+x)^{2n}+(1-x)^{2n}$, where n is a positive integer, show that: **3**

$$1+\binom{2n}{2}+\binom{2n}{4}+\ldots+\binom{2n}{2n}=\frac{4^n}{2}.$$

(b) $(1+x)^{2n}+(1-x)^{2n}$

$$=\binom{2n}{0}+\binom{2n}{1}x+\binom{2n}{2}x^2+\binom{2n}{3}x^3+\ldots+\binom{2n}{2n}x^{2n}$$

$$+\left[\binom{2n}{0}-\binom{2n}{1}x+\binom{2n}{2}x^2-\binom{2n}{3}x^3+\ldots+\binom{2n}{2n}x^{2n}\right]$$

① for expanding correctly.

$$=2\left[\binom{2n}{0}+\binom{2n}{2}x^2+\binom{2n}{4}x^4+\ldots+\binom{2n}{2n}x^{2n}\right]$$

Let $x=1$:

① for substituting $x=1$

$$(1+1)^{2n}+(1-1)^{2n}=2\left[\binom{2n}{0}+\binom{2n}{2}1^2+\binom{2n}{4}1+\ldots+\binom{2n}{2n}1\right]$$

$$2^{2n}=2\left[1+\binom{2n}{2}+\binom{2n}{4}+\binom{2n}{6}+\ldots+\binom{2n}{2n}\right]$$

$$\therefore\ 1+\binom{2n}{2}+\binom{2n}{4}+\ldots+\binom{2n}{2n}=\frac{2^{2n}}{2}=\frac{4^n}{2}$$

$\boxed{\text{① for working giving required answer}}$

(c) $OABC$ is trapezium with $\overrightarrow{OA} = 3\overrightarrow{CB}$. $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

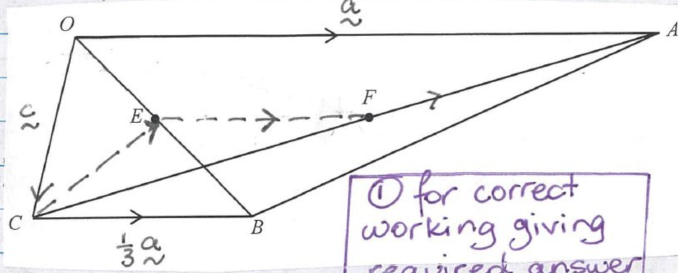
E and F are midpoints of OB and CA respectively.

(i) Show that $\overrightarrow{OE} = \frac{1}{2}\left(\underset{\sim}{c} + \frac{1}{3}\underset{\sim}{a}\right)$. **1**

(i) Given: $\overrightarrow{OA} = 3\overrightarrow{CB}$, $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OC} = \underset{\sim}{c}$

E, F are midpoints of OB, CA

Show $\overrightarrow{OE} = \frac{1}{2}\left(\underset{\sim}{c} + \frac{1}{3}\underset{\sim}{a}\right)$



$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$

$= \underset{\sim}{c} + \frac{1}{3}\underset{\sim}{a}$ ✓

$\therefore \overrightarrow{OE} = \frac{1}{2}\left(\underset{\sim}{c} + \frac{1}{3}\underset{\sim}{a}\right)$ given $OE = \frac{1}{2}CB$

① for correct working giving required answer

(ii) Find $\overrightarrow{EF}$. **3**

ii) $\overrightarrow{EF} = \overrightarrow{CF} - \overrightarrow{CE}$ where $\overrightarrow{CF} = \frac{1}{2}(\overrightarrow{OA} - \overrightarrow{OC})$

$= \frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{c})$ ✓ ① mark for $\overrightarrow{CF}$

$\overrightarrow{EF} = \frac{1}{2}(\underset{\sim}{a} - c) - \left(\frac{1}{6}\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{c}\right)$

$= \frac{1}{2}\underset{\sim}{a} - \frac{1}{2}c - \frac{1}{6}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{c}$

$= \frac{2}{6}\underset{\sim}{a}$

$= \frac{1}{3}\underset{\sim}{a}$ ✓ ① mark for $\overrightarrow{EF} = \frac{1}{3}\underset{\sim}{a}$

$\overrightarrow{CE} = \overrightarrow{OE} - \overrightarrow{OC}$

$= \frac{1}{2}\left(\underset{\sim}{c} + \frac{1}{3}\underset{\sim}{a}\right) - \underset{\sim}{c}$

$= \frac{1}{6}\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{c}$ ✓ ① mark for $\overrightarrow{CE}$

(iii) Hence, prove that $CEFB$ is a parallelogram. **1**

iii) $\overrightarrow{CB} = \frac{1}{3}\overrightarrow{OA}$ and $\overrightarrow{EF} = \frac{1}{3}\underset{\sim}{a}$ (from ii)

$= \frac{1}{3}\underset{\sim}{a}$

(given)

$\therefore$ $\overrightarrow{CB}$ is parallel to $\overrightarrow{EF}$

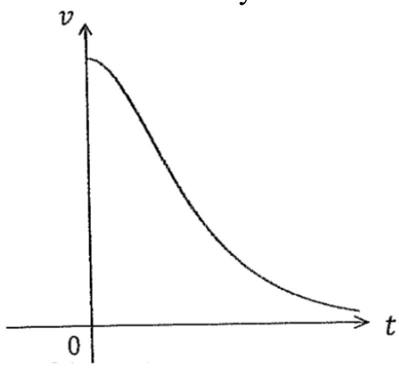
and $|\overrightarrow{EF}| = |\overrightarrow{CB}|$ ✓ ① mark for explanation

$\therefore$ CEFB is a parallelogram.

Question 14 (11 marks) Use the Question 14 Writing Booklet

(a) A particle P is moving along the x axis. At time $t=0$, it was at $x=0$. Its velocity $v\ ms^{-1}$ at a time t is given by $v=\dfrac{e^t}{1+(e^t)^2}$. **4**

The graph below shows its velocity as a function of time.



Given that the displacement of the particle when $t=k$ seconds is $\tan^{-1}\frac{1}{2}$ metres, show that $k=\ln 3$.

(a) At $t=0$, $x=0$. Also, $x=\tan^{-1}\frac{1}{2}$ when $t=k$

(4 marks)

$v=\dfrac{e^t}{1+(e^t)^2}$

$x=\displaystyle\int \frac{e^t}{1+(e^t)^2}\,dt$

$x=\tan^{-1}(e^t)+c$ ✓ $\boxed{\text{① for correct integration}}$

at $t=0$, $x=0$ $\therefore$

$0=\tan^{-1}(e^0)+c$

$0=\tan^{-1}(1)+c$

$0=\frac{\pi}{4}+c$

$\therefore c=-\frac{\pi}{4}$ ✓ $\boxed{\text{① for correct 'c'}}$

$\therefore x=\tan^{-1}(e^t)-\frac{\pi}{4}$

Now: $\tan^{-1}(\frac{1}{2})=\tan^{-1}(e^k)-\frac{\pi}{4}$

$\tan^{-1}(e^k)-\tan^{-1}(\frac{1}{2})=\frac{\pi}{4}$

$\tan\left[\tan^{-1}(e^k)-\tan^{-1}(\frac{1}{2})\right]=\tan\frac{\pi}{4}$ ✓ $\boxed{\text{① for progress}}$

using $\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\tan\frac{\pi}{4}$ $\qquad \tan\alpha=e^k$, $\tan\beta=\frac{1}{2}$

$\dfrac{e^k-\frac{1}{2}}{1+\frac{e^k}{2}}=1$

$\dfrac{2e^k-1}{2+e^k}=1$

$2e^k-1=2+e^k$

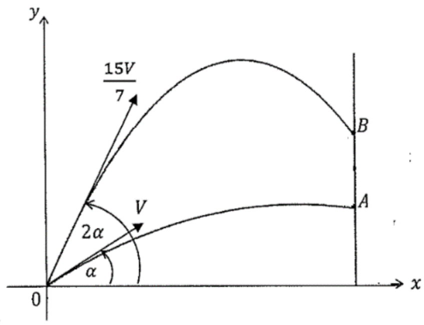
$e^k=3$ ✓ $\boxed{\text{① for correct working giving } e^k=3}$

i.e. $\log_e 3=k$

i.e. $k=\ln 3$.

(b) Jacob hits two balls at the same time from the origin O towards a target on a vertical wall.

The velocity of the first ball is $V\,ms^{-1}$ and its angle of projection is α to the horizontal while the velocity of the second ball is $\frac{15V}{7}\,ms^{-1}$ and its angle of projection to the horizontal is 2α.



(i) Show that the position vector of the second ball t seconds after be projected is given by: **3**

$$\underset{\sim}{r_2}(t) = \begin{pmatrix} \frac{15V}{7} t\cos 2\alpha \\ -\frac{1}{2}gt^2 + \frac{15V}{7} t\sin 2\alpha \end{pmatrix}$$

i) 2nd ball: (3 marks)

Horizontally

$\ddot{\underset{\sim}{x}} = 0\underset{\sim}{i}$

$\dot{x} = \underset{\sim}{c_1}$

at $t=0$, $\dot{x} = \frac{15V}{7}\cos 2\alpha\, \underset{\sim}{i}$

$\therefore \underset{\sim}{c_1} = \frac{15V}{7}\cos 2\alpha\, \underset{\sim}{i}$

$\therefore \dot{\underset{\sim}{x}} = \frac{15V}{7}\cos 2\alpha\, \underset{\sim}{i}$

$\underset{\sim}{x} = \frac{15Vt}{7}\cos 2\alpha\, \underset{\sim}{i} + \underset{\sim 3}{c}$

at $t=0$, $x=0$ $\therefore \underset{\sim 3}{c} = 0$

$\therefore \underset{\sim}{x} = \left(\frac{15V}{7} t\cos 2\alpha\right)\underset{\sim}{i}$

Vertically

$\ddot{\underset{\sim}{y}} = -g\underset{\sim}{j}$ ✓

$\boxed{\text{① for acceleration vectors}}$

$\dot{\underset{\sim}{y}} = -gt\underset{\sim}{j} + \underset{\sim}{c_2}$

at $t=0$, $\dot{\underset{\sim}{y}} = \frac{15V}{7}\sin 2\alpha\, \underset{\sim}{j}$

$\underset{\sim}{c_2} = \frac{15V}{7}\sin 2\alpha\, \underset{\sim}{j}$

$\boxed{\text{① for deriving velocity vectors}}$

$\therefore \dot{\underset{\sim}{y}} = \left(-gt + \frac{15V}{7}\sin 2\alpha\right)\underset{\sim}{j}$ ✓

$\underset{\sim}{y} = \left(-\frac{gt^2}{2} + \frac{15V}{7}t\sin 2\alpha\right)\underset{\sim}{j} + \underset{\sim}{c_4}$

at $t=0$ $y=0$ $\therefore \underset{\sim 4}{c} = 0$

$\therefore \underset{\sim}{y} = \left(-\frac{gt^2}{2} + \frac{15V}{7}t\sin 2\alpha\right)\underset{\sim}{j}$ ✓

i.e. $\underset{\sim 2}{r}(t) = \begin{bmatrix} \frac{15V}{7} t\cos 2\alpha \\ -\frac{gt^2}{2} + \frac{15V}{7} t\sin 2\alpha \end{bmatrix}$

$\boxed{\text{① for deriving position vectors}}$

(ii) It is known that both balls hit the wall after T seconds. Show that $\cos\alpha = \frac{5}{6}$. **4**

Ball 1:

Considering only horizontal motion:

$\ddot{\underset{\sim}{x}} = 0\underset{\sim}{i}$

$\dot{\underset{\sim}{x}} = \underset{\sim}{c}_5$

at $t=0$, $\dot{\underset{\sim}{x}} = V\cos\alpha\,\underset{\sim}{i}$

$\therefore \underset{\sim}{c}_5 = V\cos\alpha\,\underset{\sim}{i}$

$\therefore \dot{\underset{\sim}{x}} = V\cos\alpha\,\underset{\sim}{i}$

$\underset{\sim}{x} = Vt\cos\alpha\,\underset{\sim}{i} + \underset{\sim}{c}_6$

at $t=0$, $x=0$ $\therefore \underset{\sim}{c}_6 = 0$

$\therefore \underset{\sim}{x} = Vt\cos\alpha\,\underset{\sim}{i}$ ✓

① for correct horizontal displacement vector for ball 1. Student does not need to derive this. Can just say: 'similarly $\underset{\sim}{x} = vt\cos\alpha\,\underset{\sim}{i}$ for ball 1'

After T seconds:

$VT\cos\alpha = \frac{15V}{7}T\cos 2\alpha$

① for equating horizontal displacements

$7\cancel{V}\cancel{T}\cos\alpha = 15\cancel{V}\cancel{T}(2\cos^2\alpha - 1)$

$7\cos\alpha = 30\cos^2\alpha - 15$

$30\cos^2\alpha - 7\cos\alpha - 15 = 0$

① for equation in terms of $\cos\alpha$

$\cos\alpha = \frac{7 \pm \sqrt{49 - 4(30)(-15)}}{60}$

$\cos\alpha = \frac{7 \pm 43}{60}$

$\cos\alpha = \frac{50}{60}$, $\cos\alpha > 0$

$\therefore \cos\alpha = \frac{5}{6}$.

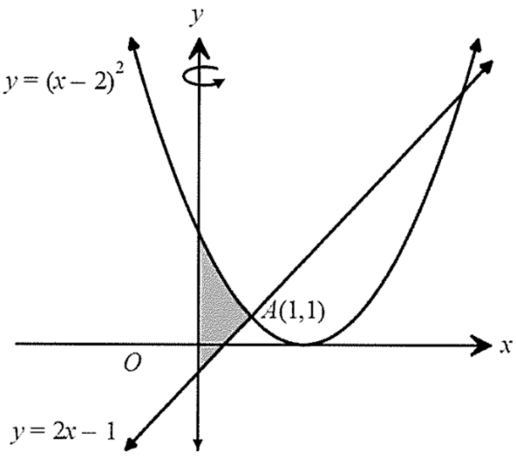
✓ ① for solving for $\cos\alpha = \frac{5}{6}$. Must show solving steps

End of Question 14

Question 15 (11 marks) Use the Question 15 Writing Booklet

(a) Consider the shaded region bounded by the line $y = 2x - 1$, the curve $y = (x-2)^2$ and the y-axis. **4**

Find the volume of the solid of revolution formed by rotating this region about the y-axis.



(a) Divide the shaded region into two parts through the line $y=1$. When the region below this line is rotated about the y-axis we obtain a cone of radius 1 and height 2.

(4 marks)

$\therefore V = \frac{1}{3}\pi(1)^2(2) + \pi\int_1^4 x^2\,dy$

$y = (x-2)^2$

$x - 2 = \pm\sqrt{y}$

$x = 2 \pm \sqrt{y}$

For the shaded area,

$x = 2 - \sqrt{y}$

$x^2 = 4 + y - 4\sqrt{y}$

$= \frac{2\pi}{3} + \pi\int_1^4 (4 + y - 4\sqrt{y})\,dy$ $\boxed{\text{1 for integral that gives volume}}$

$= \frac{2\pi}{3} + \pi\left[4y + \frac{y^2}{2} - 4.\frac{2}{3}y^{3/2}\right]_1^4$ ✓ $\boxed{\text{1 for integrating correctly}}$

$= \frac{2\pi}{3} + \pi\left[\left(16 + 8 - \frac{64}{3}\right) - \left(4 + \frac{1}{2} - \frac{8}{3}\right)\right]$ ✓ $\boxed{\text{1 for substitution}}$

$\frac{8\pi}{3} \quad - \frac{11\pi}{6}$

$= \frac{2\pi}{3} + \frac{5\pi}{6}$

$= \frac{3\pi}{2}$ units^3. ✓ $\boxed{\text{1 for answer}}$

OR

$V = \frac{\pi}{4}\int_{-1}^{1}(y+1)^2\,dy + \pi\int_1^4 (4 + y - 4y^{1/2})\,dy$ ✓ $y = 2x - 1 \rightarrow x = \frac{y+1}{2}$

$= \frac{\pi}{4}\left[\frac{(y+1)^3}{3}\right]_{-1}^{1} + \pi\left[4y + \frac{y^2}{2} - 4.\frac{2}{3}y^{3/2}\right]_1^4$ ✓

$= \frac{\pi}{4}\left[\frac{8}{3} - 0\right] + \pi\left[\left(16 + 8 - \frac{64}{3}\right) - \left(4 + \frac{1}{2} - \frac{8}{3}\right)\right]$ ✓

$= \frac{2\pi}{3} + \frac{5\pi}{6}$

$= \frac{3\pi}{2}$ units^3. ✓

(b) (i) Show that the solution of $\cos 3x - \sin 2x = 0$, for $0 < x < \frac{\pi}{2}$ is given by **3**

$$\sin x = \frac{\sqrt{5}-1}{4}.$$

You may use the identity $\cos 3x = 4\cos^3 x - 3\cos x$. DO NOT PROVE THIS.

b) i) $\cos 3x - \sin 2x = 0, \ 0 < x < \frac{\pi}{2}$

$4\cos^3 x - 3\cos x - \sin 2x = 0$

$4\cos^3 x - 3\cos x - 2\sin x \cos x = 0$

$\cos x(4\cos^2 x - 2\sin x - 3) = 0$

$\cos x(4(1-\sin^2 x) - 2\sin x - 3) = 0$

$-\cos x(4\sin^2 x + 2\sin x - 1) = 0$ ✓ $\boxed{\text{① for progress}}$

Now $\cos x = 0$ when $x = \frac{\pi}{2}$, which is not in domain given, $0 < x < \frac{\pi}{2}$ ✓ $\boxed{\text{① for why } x \neq \frac{\pi}{2}}$

$4\sin^2 x + 2\sin x - 1 = 0$

$\sin x = \dfrac{-2 \pm \sqrt{4 - 4(4)(-1)}}{8}$

$\sin x = \dfrac{-2 \pm \sqrt{20}}{8} \quad (\sqrt{20} = 2\sqrt{5})$

$\sin x = \dfrac{-1 \pm \sqrt{5}}{4}$

$\sin x = \dfrac{\sqrt{5} - 1}{4}$ as $\sin x > 0$ in domain $0 < x < \frac{\pi}{2}$ ✓

① for working giving required value for $\sin x$. Do not deduct marks twice for not stating restrictions

(ii) Without a calculator, verify that $x = \frac{\pi}{10}$ is a solution to $\cos 3x = \sin 2x$. **1**

b)(ii) $\cos 3x = \sin 2x$

If $x = \frac{\pi}{10}$, $\cos 3x$

$= \cos \frac{3\pi}{10}$

$= \sin\left(\frac{\pi}{2} - \frac{3\pi}{10}\right)$ ✓ ① mark 'SHOW' question

$= \sin \frac{2\pi}{10}$

$\therefore x = \frac{\pi}{10}$ is a solution

to $\cos 3x = \sin 2x$

(iii) Using the results obtained in parts (i) and (ii) prove that $\sin\frac{\pi}{5}\cos\frac{\pi}{10} = \frac{\sqrt{5}}{4}$. **3**

END OF EXAMINATION

(iii) $\sin\frac{\pi}{5}\cos\frac{\pi}{10}$

$= \sin\frac{2\pi}{10}\cos\frac{\pi}{10}$

$= 2\sin\frac{\pi}{10}\cos\frac{\pi}{10}\cos\frac{\pi}{10}$

either ✓ ① mark for recognising use of double ∠ formula

$= 2\sin\frac{\pi}{10}\cos^2\frac{\pi}{10}$

$= 2\sin\frac{\pi}{10}\left(1 - \sin^2\frac{\pi}{10}\right)$

$= 2\left(\frac{\sqrt{5}-1}{4}\right)\left(1 - \left(\frac{\sqrt{5}-1}{4}\right)^2\right)$ ✓ ① using (i) & (ii)

$= \left(\frac{\sqrt{5}-1}{2}\right)\left(\frac{16 - (5 - 2\sqrt{5} + 1)}{16}\right)$

$= \left(\frac{\sqrt{5}-1}{2}\right)\left(\frac{10 + 2\sqrt{5}}{16}\right)$

$= \left(\frac{\sqrt{5}-1}{2}\right)\left(\frac{5 + \sqrt{5}}{8}\right)$

$= \frac{5\sqrt{5} + 5 - 5 - \sqrt{5}}{16}$

$= \frac{\cancel{4}^{1}\sqrt{5}}{\cancel{16}_{4}}$ ✓ ① mark for working leading to required answer.

$= \frac{\sqrt{5}}{4}$

i.e. $\sin\frac{\pi}{6}\cos\frac{\pi}{10} = \frac{\sqrt{5}}{4}$.